Vibrational Measurement and Analysis using a Cantilever Beam

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Vibrational responses of a cantilever beam are measured and analyzed using accelerometers, a vibration table, and a data acquisition system. We see how a sine wave differs from a square wave and the frequency make up of both waves using a Fourier transform of the waves, which displays the incurring frequencies and their respective amplitudes. We then use accelerometers at two locations, one at the end of an aluminum beam and one at the node of the 2nd mode of the beam to model a one degree of freedom system and obtain much information from this such as damping properties of the beam and the effect the accuracy of the Fast Fourier transform with respect to the theoretical calculation of the frequency of the beam based on its geometric and material properties. Lastly, we excite the beam at different frequencies and measure the amplitude of the wave of the response form the beam. We notice a peak in amplitude at 13.5 Hz, signaling to us that resonance occurs at 13.5 Hz. We also obtain the amplitude ratio through this data and observe that it is highest at resonance because the excitation frequency highly matches the natural frequency of the beam.

INTRODUCTION

In today's world of complex innovations, vibrational analysis is essential to the majority of mechanisms. The purpose of analyzing vibrations is to see the types of loads acting on structures and to verify mathematical models to ensure that cracks and damages are prevented and/or detected immediately. To analyze vibrations in the modern world, we use sensors to measure various things and convert these quantities to physical quantities of interest. In this lab, we use accelerometers, which generate voltage signals that are proportional to the acceleration they undergo and we convert this voltage to a quantity of physical measurements.

This lab consists of 3 main parts. We begin by analyzing the responses of a sine and square wave using the input wave and response as well as the Fourier Transform of the response. Second, we analyze the response of a free vibrating cantilever beam, measuring acceleration from 2 locations of the beam, one at a node and the other at the end of the beam. Lastly, we measure the response of the beam to sinusoidal excitations at different frequencies to ascertain the damping properties of the beam. To accomplish all these tasks, we use an accelerometer, a power amplifier paired with a vibration table and aluminum alloy cantilever beam (shown together in Figure 1), and a data acquisition system.

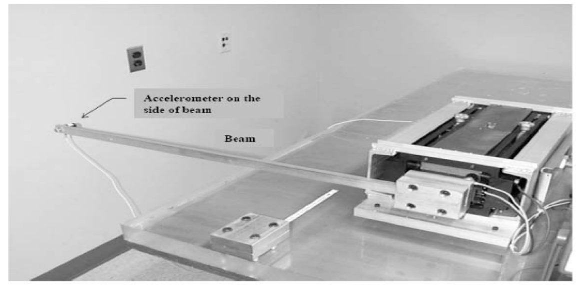

Figure 1. Vibrating Table and Beam

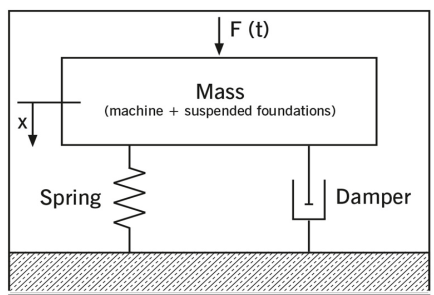

Figure 2. System with Single Degree of Freedom

The basic system we observe for vibrational analysis is that of a single degree of freedom, as shown in the diagram in Figure 1. Using the equations of motions and summing them in the diagram in Figure 2, we obtain equation 1.

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = F(t) \qquad [1]$$

Here, m is the mass of the system, c is a measure of the damping force, and k is the spring constant, and F(t) is the external force (excitation) on the system. Here, $\omega_n^2 = k/m$ and we can set $c/m = 2\zeta\omega_n$, where ζ is the damping coefficient. Solving this equation gives us

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many properties of the system, like the damping properties, which tell us the rate at which the signal will fade or not, which can be a very important quantity in many situations. In the case of this lab, we analyze an aluminum cantilever beam with dimensions 1” width x 0.5” thickness x 32” length. The vibration occurs along the horizontal axis of the width. Equation 2 gives the partial differential equation that models this behavior.

$$\rho\ddot{u}(x,t) + c(x)\dot{u}(x,t) + Lu(x,t) = 0 \quad [2]$$

In Equation 2, $\ddot{u}(x,t)$ is the deformation of the beam at location x and time t, the L operator represents the shear force in the beam, and ρ is the density of the material per unit length. Lastly, c(x) represents the damping effects present whose nature is unknown for the time being. We use a proportional damping assumption, where $c/\rho = 2\zeta\omega$. We then use separation of variables, to obtain Equation 3.

$$u(x,t) = \phi(x)q(t) \quad [3]$$

A little mathematical manipulation and substitution in the above equations results in equations 4 and 5.

$$EI\frac{d^4\varphi}{d^4x} = \omega^2\rho\varphi \quad [4]$$

$$\ddot{q}(t) + 2\zeta\,\omega\,\dot{q}\,(t) + \omega^2 q(t) = 0 \quad [5]$$

We now apply the necessary boundary and initial conditions to Equations 4 and 5. Equation 4 yields an infinite set of independent eigenfunctions or modes expressed in Equations 6, 7, 8, and 9.

$$u(x,t) = \sum_{r=1}^{\infty} \varphi_r(x)\, q_r(t) \quad [6]$$

$$\phi_r(x) = \sin\beta_r x - \sinh\beta_r x + a_r(\cos\beta_r x - \cosh\beta_r x), \quad [7]$$

$$\omega_r = (\beta_r L)^2\sqrt{EI/\rho L^4},\ r = 1, 2, ... \quad [8]$$

$$a_r = \frac{\cos\beta_r L + \cosh\beta_r L}{\sin\beta_r L - \sinh\beta_r L}\ r = 1, 2, ... \quad [9]$$

Next, we assume that we are in an underdamped system, where ($\zeta^2<1$) and thus obtain Equation 10, where C_r is related to the initial displacement, θ represents the phase angle and $q_r(t)$ are the response functions where r represents the mode that it is for. Equation 11 gives the eigenvalues for the first 3 modes of the fixed free beam.

$$q_r(t) = C_r e^{-\zeta_r\omega_r t}\sin(\omega_r\sqrt{1-\zeta_r^2}\,t + \theta) \quad [10]$$

$$\beta_1 L = 1.875 \quad \beta_2 L = 4.694 \quad \beta_3 L = 7.855 \quad [11]$$

These eigenfunctions are also called shape functions because they determine the shape of the vibrating beam. The nth mode has n-1 crossings in the beam shape. In this lab, for the beam we use, two modes are sufficient for an accurate enough analysis.

Next, we place one accelerometer at position x = x_1. Equation 12 gives the acceleration of the beam at this point. As you go along the length of the beam, you will get different contributions from the first and second modes. We then find the zero crossing point location of the second mode by setting equation 7 equal to 0, setting r = 2 and solving. We then get Equations 13.

$$\ddot{u}(x_1,t) = \phi_1(x_1)\ddot{q}_1(t) + \phi_2(x_1)\ddot{q}_2(t) \quad [12]$$

$$\varphi_2(x_1) \approx 0 \quad \text{leading to} \quad \ddot{u}(x_1,t) \approx \varphi_1(x_1)\ddot{q}_1(t) \quad [13]$$

We thus eliminate the effects of the second mode and essential make this a single degree of freedom problem, which is much easier to solve. It can also be shown that since the beam we are considering is only slightly damped, we can show by differentiating equation 9 to show that the acceleration at each beam location is proportional to the displacement at that point.

Next, to understand how a Fourier transform works, it is important to understand its use. One uses a Fourier transform to extract data about the frequency content of data sample. This is high importance in vibrational analysis because it lets you see what kind of response you are getting, a natural and stable or varied and mixed frequency response. Equation 14 is used as the basis of the Fourier Transform along with many advanced computer algorithms to get an accurate account of the frequency spectrum.

$$Z(i\omega) \approx \int_{-\infty}^{\infty} z(t)e^{i\omega t}dt \qquad [14]$$

The last part of the lab deals with a harmonic excitation, $f(t) = Re\{Ae^{i\omega t}\}$, which alters equation 5 to be Equation 15.

$$\ddot{q}_r(t) + 2\zeta_r\omega_r\dot{q}_r(t) + \omega_r^2 q_r(t) = A_r e^{i\omega t} \quad r = 1, 2, ... \qquad [15]$$

The coefficients A_r depend on the magnitude of the force f and the location at which the excitation is done. To get a solution to this equation, we assume a solution of the form, $q_r(t) = B_r e^{i\omega t}$ and obtain Equations 16 and 17.

$$B_r = A_r \frac{G_r(i\omega)}{\omega_r^2} \quad G_r(i\omega) = \frac{1}{[1-(\omega/\omega_r)^2 + 2i\zeta_r\omega/\omega_r]} \qquad [16]$$

$$|G_r(i\omega)| = \frac{1}{[(1-(\omega/\omega_r)^2)^2 + (2\zeta_r\omega/\omega_r)^2]^{1/2}} \qquad [17]$$

RESULTS AND DISCUSSION

We begin the lab by recording sine and square waves using the function generator. Using a Fourier transform, we obtain the frequency spectra for the sinusoidal and square wave, shown in Figure 3.

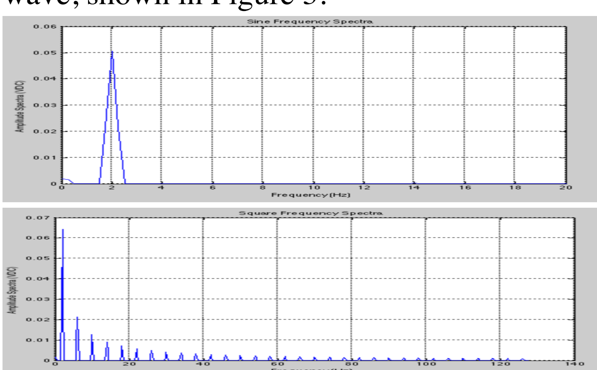

Figure 3. Sine (top) and Square (bottom) Wave Frequency Spectra.

Figure 3 makes the difference in the frequency makeup between the two graphs very obvious. The sine wave, as expected only consists of one frequency, which was the input frequency of the waveform. On the other hand, the square wave is made up of many frequencies, with the highest amplitude being that of the lowest frequency and the amplitude lessens as the frequency increases. We now recreate the sine and square waveforms using this information from the spectra. For the sine wave, the peak amplitude is 0.05 VDC and the frequency is 2 Hz. The result of the sine graph is shown in Figure 4. The two waveforms are identical in amplitude and frequency to the naked eye. The only difference is that at time t=0 the reconstructed waveform starts at voltage 0, whereas the measured sine graph starts at a positive voltage of about 0.03 VDC. Likely, this may be due to the signal generator not being set to 0 at the start of the experiment.

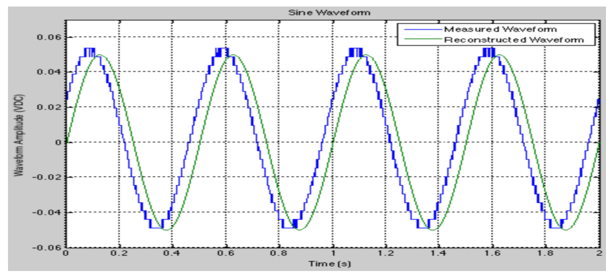

Figure 4. Sine Measured Waveform and reconstructed waveform

For the square wave, Table 1 shows the frequencies and associated amplitudes from the frequency spectra.

To show the impact of the different frequencies, we plot the response using only the first peak, the first 5 peaks, and all the peaks and show it in Figure 5. The figure shows that as the number of frequency peaks used increases, the graph looks more and more like the square graph we measured.

Table 1. Square Wave Freq. and Amplitudes

Frequency (Hz)	Amplitude (VDC)	Frequency (Hz)	Amplitude (VDC)
2	0.0644	46	0.002776
6	0.0215	50	0.002567
10	0.0129	54	0.002328
14	0.009176	58	0.002242
18	0.007127	62	0.00212
22	0.005824	66	0.001971
26	0.004982	70	0.001811
30	0.004311	74	0.00177
34	0.003762	78	0.001594
38	0.003391	82	0.001559
42	0.003096		

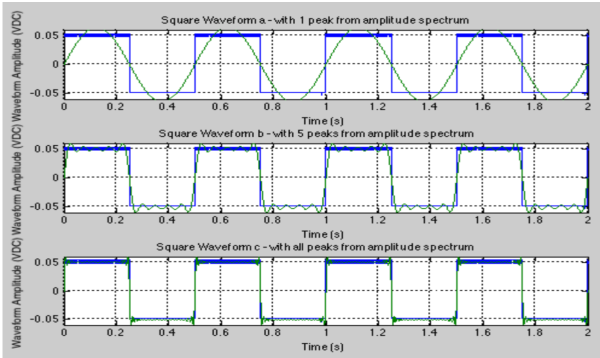

Figure 5. Square Wave and Reconstructed Waves using 1, 5, and all peaks

In the next part of the lab, we obtain the free vibrating response of the cantilever beam from an excitation at two locations of the beam: at the node of the 2nd mode, and at the end of the beam. The waveform and spectra at the two locations are shown in Figures 6 and 7. The two waveform graphs are similar in shape, but the amplitude at location 2 is nearly double that at location 1 because it was probably deflected more initially. It seems like the damped effect is more prevalent at location 2 than 1, but this may be just due to the higher initial deflection. The frequency amplitude spectra show that at location 1, there is only one frequency whereas at location 2 there are two frequencies. This is because at location 1 there is only 1 mode acting on that location because the second mode's crossing is at that location. For the end of the beam, both modes contribute to the response.

Using Equation 8 and 11, we calculate the theoretical values of the natural frequency and convert it to Hz by dividing by 2π (Calculations in Appendix). We obtain f_1 = 0.79855 Hz and f_2 = 5.005 Hz. We compare these values with the measured wave by obtaining the time between 7 peaks and taking the average to get the approximate period T. Using the relationship f = 1/T, we get the measured frequency to be 16.67 Hz.

Table 2 shows the values of the theoretical frequencies calculated using equations 8 and 11, and the material properties of the aluminum beam used. It also shows the frequencies read from the frequency spectrums at each location and the average frequency

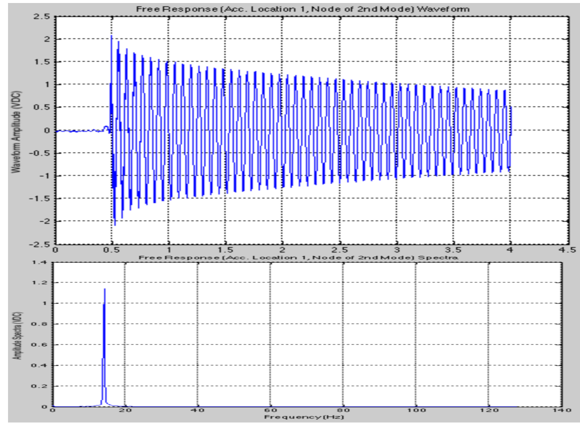

Figure 6. Free Response Location 1 (Node of 2nd Mode) Wave (top) & Spectra (bottom)

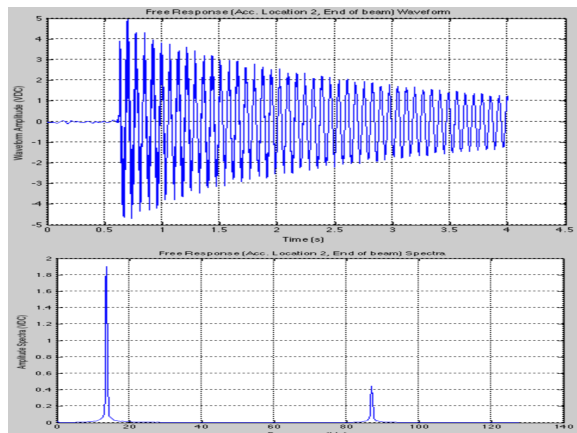

Figure 7. Free Response Location 2 (End of Beam) Wave (top) & Spectra (bottom)

obtained from the period of the 1st location waveform. The % difference for the average is quite low at around 5% response. The freq.'s

Table 2. Measured vs. Theoretical Frequencies

	F (Hz)	% Difference
Theoretical w_1	15.87	-
Theoretical w_2	99.48	-
Average w_1	16.6667	5.041
Spectrum w_1	14.2460	10.233
Spectrum w_2	86.9754	12.570

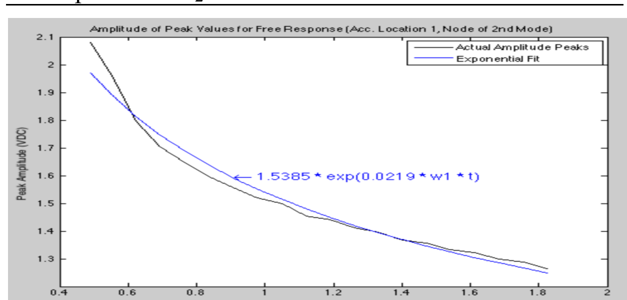

Figure 8. Amplitude Peak Curve

Next, we obtain the amplitude of the peaks in the waveform of the free response at location 1 and plot it as shown in Figure 8. As expected, the curve looks like an exponential decay, which is consistent with the damping we can expect in the cantilever beam. Next, we make an exponential fit of the data and obtain an exponential equation for the curve of the form $De^{-\zeta \omega_i t}$ whose values are shown in table 3. Next, we use Equation 18 to get what is called the logarithmic decrement, from which we can approximate the damping factor as $\zeta \cong \delta/2\pi$ by assuming small values for δ. The results of the first 10 peaks are in Table 3.

$$\delta = ln\frac{q(t_1)}{q(t_2)} = \zeta\omega_1 T = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \quad [18]$$

There are many reasons why the logarithmic decrement is not the same throughout response. The weight of the accelerometer may affect the damping behavior as well as the exactness of the location at which it is placed relative to the node of the 2nd mode. Another way of identifying the damping properties is by doing a linear fit of the amplitude curve and comparing that.

Table 3. Log Decrement and Damping Factor

Peak #	$q(t_s)$	T	δ (btw t_s & t_{s+1})	ζ
1	2.0800	0.0623	0.0604	0.0096
2	1.9580	0.0713	0.0859	0.0137
3	1.7969	0.0708	0.0530	0.0084
4	1.7041	0.0708	0.0350	0.0056
5	1.6455	0.0708	0.0299	0.0048
6	1.5970	0.0710	0.0250	0.0040
7	1.5576	0.0706	0.0254	0.0040
8	1.5186	0.0713	0.0130	0.0021
9	1.4990	0.0703	0.0297	0.0047
10	1.4551	0.0708	0.0097	0.0015
11	1.4410	0.0705	-	-
12	1.4112	0.0705	-	-
13	1.3965	0.0700	-	-
14	1.3672	0.0705	-	-
15	1.3574	0.0703	-	-
16	1.3330	0.0706	-	-
17	1.3232	0.0707	-	-
18	1.2988	0.0706	-	-
19	1.2891	0.0708	-	-
20	1.2647	-	-	-
Curve	**Fit**		**D=1.5385**	**ζ=0.0691**

Finally, for the last part of lab we look at resonance by exciting the beam with forces of varying frequencies. By observing where the peak in amplitude occurs, we see at what frequency resonance occurs. Table 4 shows the input and output amplitudes for all the frequencies measured. The peak is clearly at a frequency of 13.5 Hz, which is the natural frequency of the beam under harmonic excitation.

Table 4. Input & Output Amplitude Peaks

F (Hz)	$A_{input}(f)$ (VDC)	$A_{output}(f)$ (VDC)
2.0000	0.2565	0.0612
2.5000	0.3337	0.0815
3.0000	0.4056	0.1014
3.5000	0.4727	0.1216
4.0000	0.5321	0.1417
4.5000	0.5824	0.1611
5.0000	0.6215	0.1797
5.5000	0.6462	0.1966
6.0000	0.6641	0.2135
6.5000	0.6713	0.2301
7.0000	0.6699	0.2464
7.5000	0.6622	0.2632
8.0000	0.6486	0.2817
8.5000	0.6284	0.3016
9.0000	0.6021	0.3241
9.5000	0.5728	0.3519
10.0000	0.5413	0.3895
10.5000	0.5026	0.4416
11.0000	0.4653	0.4961
11.5000	0.4032	0.5986
12.0000	0.3141	0.7649
12.5000	0.1745	1.0900
13.0000	0.1849	1.6993
13.5000	1.1109	3.1882
14.0000	1.3514	2.3571
14.5000	1.0289	1.2989
15.0000	0.8740	0.8787
15.5000	0.7872	0.6737
16.0000	0.7231	0.5330
16.5000	0.6769	0.4392
17.0000	0.6424	0.3718
17.5000	0.6133	0.3186
18.0000	0.5889	0.2778
18.5000	0.5710	0.2475
19.0000	0.5519	0.2215
19.5000	0.5371	0.2009
20.0000	0.5244	0.1840
30.0000	0.3603	0.0877
40.0000	0.2664	0.0508
50.0000	0.2215	0.0461

Furthermore, we notice that the input amplitude is not constant throughout the frequencies. It varies at almost all the frequencies in our measurements. This may be because as the frequency is changed, it takes time for the amplitude to become steady from the change, which is not accounted for in the quick transitions in frequencies.

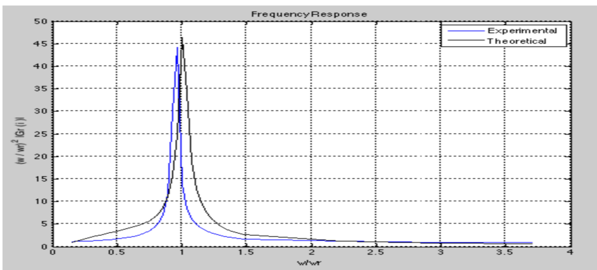

Figure 9. Frequency Response of 2nd derivative

We non-dimenionalize the experimental amplitude ratio using Equation 19, plot it in Figure 9 and compare it to the estimated theoretical curve using the damping coefficient. The theoretical curve's peak is exactly at the unity ratio, meaning, whereas the experimental is a little to the left of it. Otherwise, the ratios are quite similar in shape, with the theoretical being only a little wider.

$$\frac{A(f)_{output}\; A(50Hz)_{input}}{A(50Hz)_{output}\; A(f)_{input}} \qquad [19]$$

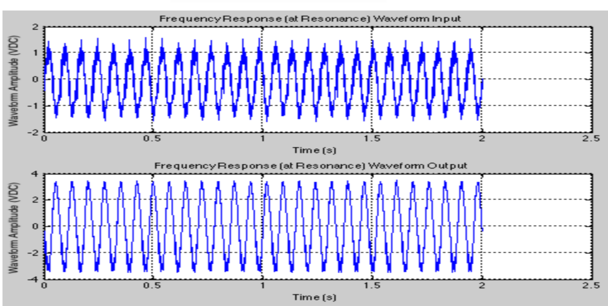

Figure 10. Free Response Resonance Wave

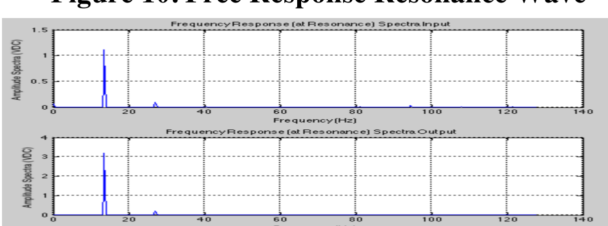

Figure 11. Free Response Resonance Spectra

Lastly, we can see from Figures 11 that there are two main dictating modes by noting the two large frequencies. From Figure 10, we see that at resonance, there is no damping and the response is constant and non-fading. As you can see from the excitation and response curves in Figure 10, the two waves are exactly out of phase. This tells that there is a 180 degree phase lag between the two and thus doubles the amplitude of the wave.

Some of the sources of error that could have arisen in this lab are in instrumentation error, errors in movement from the room, such are pushing the vibration table, and other aerodynamic forces such as wind that could distort the frequency/amplitude of the beam vibration. In general, the results were very close to what was expected based on theory.

CONCLUSIONS

In this lab we learn how to make vibrational measurements in a digital environment for accurate measurements and analysis. We learned how to use vibrational equations to our advantage especially in the second part of the lab to simplify the system. We see that a square wave is made of multiple sinusoidal waves of different frequencies of different amplitudes as seen by the Fourier transform frequency spectra and a reconstruction of the wave. Next, we see that we can model a one degree of freedom system by measuring acceleration at the node of a mode of the beams vibration by placing an accelerometer there and thus singling out the other mode's effect on the beam's vibration. We obtain the theoretical frequency of the beam using the many equations mentioned and compare it to measured frequencies using the spectra and an average using the waveform itself. Lastly, we determine the resonance frequency by getting the spectral peaks at different frequencies and seeing at which frequency the largest amplitude occurs. We also learn how to analyze this resonance behavior and non-dimensionalize it to compare with other data.

REFERENCES

[1] Drazer, German, *Vibration Measurements and Analysis*, Rutgers University, New Brunswick,2017

APPENDIX

MatLab code:

```
% Part 1
% sine wave
y = @(t) 0.05*sin(2*2*pi*t);

y1 = @(t) 0.0644*sin(2*2*pi*t);

y5 = @(t) 0.0644*sin(2*2*pi*t) +
0.0215*sin(6*2*pi*t) +
0.0129*sin(10*2*pi*t) +
0.009176*sin(14*2*pi*t) +
0.007127*sin(18*2*pi*t);

yall = @(t)
0.0644*sin(2*2*pi*t) +
0.0215*sin(6*2*pi*t) +
0.0129*sin(10*2*pi*t) +
0.009176*sin(14*2*pi*t) +
0.007127*sin(18*2*pi*t) +
0.005824*sin(22*2*pi*t) +
0.004982*sin(26*2*pi*t) +
0.004311*sin(30*2*pi*t) +
0.003762*sin(34*2*pi*t) +
0.003391*sin(38*2*pi*t) +
0.003096*sin(42*2*pi*t) +
0.002776*sin(46*2*pi*t) +
0.002567*sin(50*2*pi*t) +
0.002328*sin(54*2*pi*t) +
0.002242*sin(58*2*pi*t) +
0.00212*sin(62*2*pi*t) +
0.001971*sin(66*2*pi*t) +
0.001811*sin(70*2*pi*t) +
0.00177*sin(74*2*pi*t) +
0.001594*sin(78*2*pi*t) +
0.001559*sin(82*2*pi*t);

% Part 2
% time and amplitude of first 20 peaks:
time = [0.49 0.5523 0.6236 0.6944 0.7652 0.836
0.907 0.9776 1.0489 1.1192 1.19 1.2605 1.331
1.401 1.4715 1.5418 1.6124 1.6831 1.7537 1.8245];
amp = [2.08 1.958 1.7969 1.7041 1.6455 1.597
1.5576 1.5186 1.499 1.4551 1.441 1.4112 1.3965
1.3672 1.3574 1.333 1.3232 1.2988 1.2891 1.2647];

% Exponential fit of amplitude curve
p = polyfit(log(time),log(amp),1)  % Exponential fit
m = p(1)
b = exp(p(2))
fit = @(time) b*time.^m
% Obtain delta and T using equation
for i = 1:9
   delta(i) = log(amp(i)/amp(i+1));
   T(i) = time(i+1)-time(i);
end
```

Natural Frequency Calulation:
E = 10000 ksi
I = 1/12w*t^3 = 1/12 * 1 * 0.5^3
Rho = d * A = 9.754369 * 10^-2
Beta1L = 1.875
Beta2L = 4.694
Using equation 8:
w_r = ((betaL)^2 * sqrt(EI/(rho*L)))*1/2pi [Hz]
We get:
w_1 = 15.87 Hz
w_2 = 99.48 Hz

```
% Damping factor calculation.
w1 =  15.87
zeta = delta/(2*pi)
zeta1 = - m / w1

% Part 3
% frSP is the matrix obtained in Part 3 of the
% Spectral Peak

table = [frSP(:,1) frSP(:,3) frSP(:,5)]
A50in = table(end,2)
A50out = table(end,3)
ratio = (table(:,3)*A50in)./(table(:,2)*A50out)
wwr = table(:,1)/13.5
```